Gauge Dependence of Four-Fermion QED Green Function and Atom-Like Bound State Calculations

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Abstract

We derive a reation between four-fermion QED Green functions of different covariant gauges which defines the gauge dependence completely. We use the derived gauge dependence to check the gauge invariance of atom-like bound state calculations. We find that the existing QED procedure does not provide gauge invariant binding energies. A way to a corrected gauge invariant procedure is pointed out.

1 Introduction

The persisting discrepancy between theory and experiment for positronium width [1] is a chalenge for QED. At the moment the hope is on taking into account corrections of relative order α^2 [2, 3]. In the circumstances the question of self-consistency of the calculations, in particular, of gauge invariance of the result is of prime concern.

The modern way to calculate parameters of two-particle atom-like bound states is to extract them from corresponding four-fermion QED Green function (see, for example, [4, 5, 6] and this paper below). Thus, to check the gauge independence of the calculated bound state parameters, one should carry the gauge parameter through all the extraction procedure. (An example of this see in [7] where the gauge independence of the correction to the positronium width of relative order α was checked.) The extraction procedure gets more and more complicated with an increase in order of radiative corrections and direct order by order check of gauge invariance becomes impractical as a check of self-consistency of the calculations. Instead, one would like to exploit gauge invariance choosing a most convenient gauge and switching from one gauge to another in the process of the calculations. In view of these complications, it seems pertinent to make a step out of the concrete practice of bound-state calculations and to study first the gauge dependence of the four-fermion QED Green function itself without taking into account the complications of the bound-state parameter calculations.

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In the present paper we derive a relation between four-fermion QED Green functions of different values of gauge-fixing parameter (we consider the covariant gauges only). The relation completely defines the evolution of the Green function in the gauge-fixing parameter. Our derivation does not use perturbation theory. Next, we use our relation to check gauge invariance of the extraction procedure of atom-like bound-state parameters. The result is negative. It turns out that the existing procedure provides gauge-dependent answers for binding energies. We find a flaw in the procedure which is responsible for the gauge-dependence of the result and point the way to its correction.

Next section contains a derivation of the evolution in the gauge-fixing parameter; section 3 comprises a brief recall of the extraction procedure and an utilization of the general evolution formula from section 2 for an analysis of gauge-dependence of the extraction; in the last, fourth, section we point out the reason for the gauge dependence and the way to the correct procedure.

2 Evolution in Gauge-Fixing Parameter

Let us consider the four-fermion QED Green function

$$G_\beta(x_f,\overline{x}_f,x_i,\overline{x}_i) \equiv i\int D\psi DA\,\exp\left(iS_{QED}(\beta)\right)(\overline{\psi}(\overline{x}_f)\psi(x_f))(\overline{\psi}(x_i)\psi(\overline{x}_i))\,, \tag{1}$$

where $x_f(\overline{x}_f)$ is a coordinate of outgoing particle (antiparticle) and $x_i(\overline{x}_i)$ is the same for ingoing pair. The definition of gauge fixing parameter β is given by corresponding photon propagator:

$$D_{\mu\nu}(\beta,x) = \int\frac{dk}{(2\pi)^4}\left(-g_{\mu\nu}+\beta\frac{k_\mu k_\nu}{k^2}\right)\frac{i}{k^2}e^{ikx}. \tag{2}$$

Our aim is to study the dependence of G_β on β. To this end, it is useful to consider a Green function in external photon field, $G(A)$, which is a result of integration over the fermion field in the rhs of (1). From the one hand, it is simply connected to the Green function [8]:

$$G_\beta = (e^{L_\beta}G(A))_{A=0}\,,\;\; L_\beta \equiv \frac{1}{2}\frac{\delta}{\delta A_\mu}D_{\mu\nu}(\beta)\frac{\delta}{\delta A_\nu}. \tag{3}$$

(In this formula each L_β generates a photon propagator; the dependence on the coordinates of ingoing and outgoing particles is suppressed for brevity.) From the other hand, $G(A)$ is siply connected to a gauge invariant object $G_{inv}(A)$:

$$G(A) = G_{inv}(A)\exp\left(ie\int_{\overline{x}_f}^{x_f}A_\mu dx^\mu - ie\int_{\overline{x}_i}^{x_i}A_\mu dx^\mu\right). \tag{4}$$

The gauge invariance of G_{inv} means that it is independent of the longitudinal component of A:

$$\partial_\mu\frac{\delta}{\delta A_\mu}G_{inv}(A) = 0 \tag{5}$$

and is a consequence of gauge invariance of the combination

$$\overline{\psi}(x)\exp\left(ie\int_y^x A_\mu dz^\mu\right)\psi(y). \tag{6}$$

A substitution of (4) into (3) yields

$$G_\beta = \left(e^{L_\beta} G_{inv}(A) \exp\left(ie \int_{\overline{x}_f}^{x_f} A_\mu dx^\mu - ie \int_{\overline{x}_i}^{x_i} A_\mu dx^\mu \right) \right)_{A=0}. \tag{7}$$

Let us take a β-derivative of both sides of this equation:

$$\frac{\partial}{\partial\beta} G_\beta = \left(e^{L_\beta} (\partial_\beta L_\beta) G_{inv}(A) \exp\left(ie \int_{\overline{x}_f}^{x_f} A_\mu dx^\mu - ie \int_{\overline{x}_i}^{x_i} A_\mu dx^\mu \right) \right)_{A=0}. \tag{8}$$

To get an evolution equation, one needs to express the rhs of this equation in terms of G_β. It is possible because $(\partial_\beta L_\beta)$ commutes with $G_{inv}(A)$ and gives a c-factor when acts on the consequent exponential. So, (8) transforms itself into

$$\frac{\partial}{\partial\beta} G_\beta(x_f, \overline{x}_f, x_i, \overline{x}_i) = F(x_f, \overline{x}_f, x_i, \overline{x}_i) G_\beta(x_f, \overline{x}_f, x_i, \overline{x}_i), \tag{9}$$

where we have restored the x-dependence and used F to denote the action of $(\partial_\beta L_\beta)$ on the exponential:

$$\begin{aligned} &(\partial_\beta L_\beta) \exp\left(ie \int_{\overline{x}_f}^{x_f} A_\mu dx^\mu - ie \int_{\overline{x}_i}^{x_i} A_\mu dx^\mu \right) \equiv \\ &\qquad F(x_f, \overline{x}_f, x_i, \overline{x}_i) \exp\left(ie \int_{\overline{x}_f}^{x_f} A_\mu dx^\mu - ie \int_{\overline{x}_i}^{x_i} A_\mu dx^\mu \right). \end{aligned} \tag{10}$$

An explanation is in order: In deriving (9) we have used a commutativity of $(\partial_\beta L_\beta)$ and $G_{inv}(A)$; it is a direct consequence of gauge invariance of G_{inv} (see (5)) and the fact that $(\partial_\beta L_\beta)$ contains only derivatives in longitudinal components of A (see (3) for a definition of L_β and (2) for β-dependence of $D_{\mu\nu}$).

The solution of eq.(9) for β-evolution is

$$G_\beta(x_f, \overline{x}_f, x_i, \overline{x}_i) = \exp\left((\beta - \beta_0) F(x_f, \overline{x}_f, x_i, \overline{x}_i) \right) G_{\beta_0}(x_f, \overline{x}_f, x_i, \overline{x}_i). \tag{11}$$

To get the final answer one needs an explicite view of F from (11). It is easily deduced from the F-definition (10) and the following representation for the longitudinal part of the photon propagator:

$$\partial_\beta D_{\mu\nu}(\beta, x) = -\frac{1}{16\pi^2} \partial_\mu \partial_\nu \ln((x^2 - i\varepsilon) m^2), \tag{12}$$

where m is an arbitrary mass scale which is fixed, for defineteness, on the fermion mass. Then, up to an additive constant,

$$F = \frac{\alpha}{4\pi} \left(\ln \frac{1}{m^4 (x_f - \overline{x}_f)^2 (x_i - \overline{x}_i)^2} + \ln \frac{(x_f - x_i)^2 (\overline{x}_f - \overline{x}_i)^2}{(x_f - \overline{x}_i)^2 (\overline{x}_f - x_i)^2} \right). \tag{13}$$

Substituting (13) into (11), we get our final aswer for β-evolution:

$$\begin{aligned} G_\beta(x_f, \overline{x}_f, x_i, \overline{x}_i) \;=\; & \left[\frac{Z (x_f - x_i)^2 (\overline{x}_f - \overline{x}_i)^2}{m^4 (x_f - \overline{x}_f)^2 (x_i - \overline{x}_i)^2 (x_f - \overline{x}_i)^2 (\overline{x}_f - x_i)^2} \right]^{\frac{\alpha}{4\pi}(\beta - \beta_0)} \times \\ & G_{\beta_0}(x_f, \overline{x}_f, x_i, \overline{x}_i). \end{aligned} \tag{14}$$

The normalization Z is infinite before the ultraviolet renormalization. After the renormalization it is scheme-dependent and calculable order by order in perturbation theory. We will not need its value in what follows.

3 The Bound State Parameters And The Four-Fermion QED Green Function

The four-fermion QED Green function contains too much information for one who just going to calculate bound-sate parameters. Ona can throw away unnessesary information by putting senter of mass space-time coordinate of ingoing pair and relative times of both ingoing and outgoing pairs to zero:

$$G_{(et)\beta}(t, \mathbf{x}, \mathbf{r}', \mathbf{r}) \equiv G_\beta\left(x_f(t, \mathbf{x}, \mathbf{r}'), \overline{x}_f(t, \mathbf{x}, \mathbf{r}'), x_i(\mathbf{r}'), \overline{x}_i(\mathbf{r}')\right), \tag{15}$$

where the space-time coordinates depend on a space-time coordinate of the center of mass of the outgoing pair $(t, \mathbf{x})$ and a relative space coordinate of outgoing $(\mathbf{r}')$ and ingoing $(\mathbf{r})$ pair. In the case of equal masses

$$\begin{aligned} x_f = (t, \mathbf{x} + \frac{\mathbf{r}'}{2}), &\qquad \overline{x}_f = (t, \mathbf{x} - \frac{\mathbf{r}'}{2}), \\ x_i = (0, \frac{\mathbf{r}}{2}), &\qquad \overline{x}_i = (0, -\frac{\mathbf{r}}{2}). \end{aligned} \tag{16}$$

$G_{(et)\beta}$ still contains an unnecessary piece of information — the dependence on the center of mass space coordinate. The natural way to remove it is to go over to momentum representation and put the center of mass momentum to zero. In coordinate representation, which is more convenient for gauge invariance check, we define the propagator D_β of the fermion pair:

$$G_{(et)\beta}(t, \mathbf{x}, \mathbf{r}', \mathbf{r}) \equiv D_\beta(t, \mathbf{r}', \mathbf{r})\delta(\mathbf{x}) + \ldots, \tag{17}$$

where dots denote terms with derivatives of $\delta(\mathbf{x})$. It is natural to consider D_β as a time dependent kernel of an operator acting on wave-functions of relative coordinate. In what follows we will not make difference between a kernel and the corresponding operator. The naturalness of the above definition of the propagator is apparent in the nonrelativistic approximation:

$$e^{i2mt} D_\beta(t) \approx \sum_{E_0} \theta(t) e^{-iE_0 t} P(E_0), \tag{18}$$

where the summation runs over the spectrum of nonrelativistic Coulomb problem and $P(E_0)$ are the projectors onto corresponding subspaces of the nonrelativistic state space. One can obtain (18) keeping leading term in α-expansion of the lhs if one will keep $t \propto 1/\alpha^2$ and $\mathbf{r}', \mathbf{r} \propto 1/\alpha$ (see [6, 9]). The subscript on E_0 is to denote that it will get radiative corrections (see below). The exponential in the lhs is to make a natural shift in energy zero. In what follows we will include the energy shift in the definition of $D_\beta(t)$.

The next step in calculation of radiative corrections to the energy levels is a crucial one: one should make an assumption about the general form of a deformation of the t-dependence of the rhs of (18) caused by relativistic corrections. A naturall guess and the one which leads to the generally accepted rules of calculation of the relativistic corrections to the energy eigenvalues (see, for example [4]) is to suppose that one can contrive oscillating part of the exact propagator D_β from the rhs of (18) just shifting energy levels and modifying the operator coefficiens $P(E_0)$:

$$D_\beta(t) = \sum_{E_0 + \Delta_{E_0}} \theta(t) e^{-i\left(E_0 + \Delta_{E_0}\right)t} P_\beta(E_0 + \Delta_{E_0}) + \ldots, \tag{19}$$

where dots denote terms which are slowly-varying in time (the natural time-scale here is $1/E_0$). The additional subscript β on P_β is to denote that oscillating part of $D_\beta(t)$ can acquire a gauge parameter dependence from relativistic corrections.

Let us see how one can use eq.(19) in energy level calculations. It is quite sufficient to consider $D_\beta(t)$ on relatively short times when $\Delta_{E_0} t \ll 1$, $E_0 t \sim 1$. For such times one can approximate D_β expanding the rhs of eq.(19) over $\Delta_{E_0} t$:

$$D_\beta(t) \approx \sum_{E_0} \theta(t) e^{-iE_0 t} \sum_k t^k A_\beta^{(k)}(E_0), \tag{20}$$

where

$$A_\beta^{(k)}(E_0) = \sum_{\Delta_{E_0}} \frac{(-i\Delta_{E_0})^k}{k!} P_\beta(E_0 + \Delta_{E_0}). \tag{21}$$

An extraction of these objects from the perturbation theory is an interim step in the level shift calculations. (Here we should mention that in calculational practice $A_\beta^{(k)}(E_0)$ are exracted in momentum representation — i.e. not as coefficients near the powers of time but as the ones near the propagator-like singularities $(E - E_0 + i\varepsilon)^{-(k+1)}$.) To come nerer to the level shift values, useful objects are

$$A_\beta^{(k)} \equiv \sum_{E_0} A_\beta^{(k)}(E_0) i^k k!. \tag{22}$$

Namely, as notations of (21) suggest, eigenvalues of $A_\beta^{(0)}$ should be equal to normalizations of bound state wave functions which are driven from unit by relativistic corrections while the eigenvalues of $A_\beta^{(k)}$ should be energy shifts to the k-th power times corresponding normalizations. Thus, the eigenvalues of

$$S_\beta^{(k)} \equiv \frac{\left[A_\beta^{(0)}\right]^{-1} A_\beta^{(k)} + A_\beta^{(k)} \left[A_\beta^{(0)}\right]^{-1}}{2} \tag{23}$$

should be just energy shifts to the k-th power. Thus, we define

$$S_\beta \equiv S_\beta^{(0)} \tag{24}$$

to be the energy shift operator: its eigenvalues are the energy level shifts caused by relativistic corrections. Our aim is now to check β-independence of S_β eigenvalues.

Some notes are in order: If the conjecture (19) is true, $A_\beta^{(0)}$ should commute with $S_\beta^{(k)}$ and the following relation should hold:

$$S_\beta^{(k)} = [S_\beta]^k \tag{25}$$

This relation was suggested as a check of the cojecture (19) in [6] and, to our knowlege, has never been checked. Another thing to note is that relativistic corrections affects the form of the scalar product of wave functions and, thus, one shoud add a definition of operator products to the formal expressions (23),(25). But the level of accuracy to which we will operate permits us not to go into this complication and use the operator products as they

are in the nonrelativistic approximation — i.e. as the convolution of the corresponding kernels.

The way to the gauge invariance check of the energy shift calculations is clear now: Using the gauge evolution relation (14) one should find the β-dependence of S_β and then of its eigenvalues. As S_β is defined in (24),(23) through $A_\beta^{(k)}$'s which are, in turn, defined in (20) through the propagator D_β, the first step is to simplify (14) to the reduced case of zero relative time and total momentum of the fermion pair:

$$\begin{aligned} D_\beta(t,\mathbf{r}',\mathbf{r}) &= \left[\frac{(1-(\mathbf{r}'-\mathbf{r})^2/(4t^2))}{(1-((\mathbf{r}'+\mathbf{r})^2/(4t^2))}\right]^{\frac{\alpha}{2\pi}(\beta-\beta_0)} \times \\ & \left[\frac{Z}{m^2\mathbf{r}'^2 m^2\mathbf{r}^2}\right]^{\frac{\alpha}{4\pi}(\beta-\beta_0)} D_{\beta_0}(t,\mathbf{r}',\mathbf{r}). \end{aligned} \tag{26}$$

The factor in the square brackets of the second line is time-independent and futher factorizible on factors depending on either ingoing or outgoing pair parameters. This reduce the influence of this factor to a change in the normalization of states. Being interested in gauge invariance of energy shifts, we omit this factor in what follows. Let us turn to the analysis of the influence of the factor in the first line of (26).

This factor is close to unit in the atomic scale $\mathbf{r}',\mathbf{r}\sim 1/\alpha$, $t\sim 1/\alpha^2$. We will use its approximate form:

$$Factor \approx 1+\frac{\alpha}{2\pi}(\beta-\beta_0)\frac{\mathbf{r}'\mathbf{r}}{t^2}+O(\alpha^5). \tag{27}$$

One can read the dependence of $A_\beta^{(k)}$ on β from (20),(26),(27) as

$$A_\beta^{(k)} \approx A_{\beta_0}^{(k)} - \frac{\alpha}{2\pi}\frac{(\beta-\beta_0)}{(k+1)(k+2)}\mathbf{r}A_{\beta_0}^{(k+2)}\mathbf{r}, \tag{28}$$

where $\mathbf{r}$ is the vector operator of relative position of interacting particles. The mixing of different $A_\beta^{(k)}$'s with a change in the gauge parameter is due to the presence of $1/t^2$ in the rhs of (27). Finally, using the definition (24), relations (25) and the fact that

$$A^{(0)} \approx 1 \tag{29}$$

in the nonrelativistic approximation one can derive the following β-dependence of S_β:

$$\begin{aligned} S_\beta \approx\ & S_{\beta_0} - \\ & \frac{\alpha}{2\pi}(\beta-\beta_0)\left(\frac{1}{6}\mathbf{r}S_{\beta_0}^3\mathbf{r}-\frac{1}{4}S_{\beta_0}\mathbf{r}S_{\beta_0}^2\mathbf{r}-\frac{1}{4}\mathbf{r}S_{\beta_0}^2\mathbf{r}S_{\beta_0}\right). \end{aligned} \tag{30}$$

Treating the term in the last line of the rhs of the above relation as a perturbation, one can get an approximate value of the β-dependent piece of the energy shift just averaging the perturbation with respect to the corresponding eigenstate of S_{β_0}.

Thus, we get for the leading order of β-derivative of an energy shift the following representation:

$$\left(\frac{\partial}{\partial\beta}\Delta_\beta\right)_L = -\frac{\alpha}{2\pi}\left(\frac{1}{6}\left\langle\mathbf{r}S_L^3\mathbf{r}\right\rangle-\frac{1}{4}\left\langle S_L\mathbf{r}S_L^2\mathbf{r}\right\rangle-\frac{1}{4}\left\langle\mathbf{r}S_L^2\mathbf{r}S_L\right\rangle\right), \tag{31}$$

where $\langle \ldots \rangle$ means averaging with respect to the corresponding nonrelativistic eigenstate and the subscript L means the leading order in α-expansion.

Eq.(31) is sufficient to define an order in α in which the energy shifts become gauge dependent:

$$\left(\frac{\partial}{\partial\beta}\Delta_\beta\right)_L \sim \alpha^{11}. \tag{32}$$

Here we have taken into account that $\mathbf{r} \sim 1/\alpha$ and $S_L \sim \alpha^4$.

To have a gauge dependence in any observable is clearly unacceptable. In the next section we will see how one should correct the above procedure of energy shift extraction from the QED Green function to get rid of the gauge dependence of energy shifts.

4 A Way Out

The procedure recalled in the previous section is based on the conjecture (19). A consequence of this conjecture is the gauge dependence of energy shifts of (31). One can conclude that the conjecture is wrong. In particular, as one can infer from eq.(26), the operator coefficients near the oscillating exponentials in (19) shoud get a time dependence from relativistic corrections. Even if in some gauge they are time independent, the gauge parameter evolution should generate a dependence which in the leading order in α reduce itself to the following replacement in (19):

$$P_\beta(E_0 + \Delta_{E_0}) \to P_\beta(E_0 + \Delta_{E_0}) + \frac{\Sigma_\beta(E_0)}{t^2}. \tag{33}$$

That $\Sigma_\beta(E_0)$ has nothing to do with energy shifts but will give contributions to $A_\beta^{(k)}(E_0)$'s from eq.(20). Being gauge dependent these contributions lead to the gauge dependence of energy shifts.

The way to the correct procedure is to through away terms like $\Sigma_\beta(E_0)/t^2$ prior to the definition of the energy shift operator. Thus, a necessary step in the process of extracting energy shifts from the QED Green function (and the one which necessity is not recognized in the stanard procedure) is to calculate and subtract contributions like the last term in the rhs of (33) from the propagator of the fermion pair.

Below we report on a calculation of $\Sigma_\beta(E_0)$ from (33). The most economical way to calculate it is to note that the energy dependence of the Fourier transform of the corresponding contribution to the propagator is

$$(E - E_0)\ln(-(E - E_0 + i\varepsilon)) \tag{34}$$

and that it comes from diagrams describing radiation and subsequent absorption of a soft photon with no change in the level E_0 of the radiating and absorbing bound state. Similar contributions (with another power of energy before the *log*) are well known for the propagator of a charged fermion [10]

The first step in our calculation is to present the pair propagator in the following form:

$$D_\beta(t) \approx \left(e^{L_s} e^{ie\mathbf{r}\mathbf{A}(t)} D_{inv}(t, A) e^{-ie\mathbf{r}\mathbf{A}(0)}\right)_{A=0}, \tag{35}$$

where L_s is the same as in (3) except a restriction on the momentum of photon propagator — the range of its variation is restricted to the soft region which border is of order of atomic binding energies; the exponentials with gauge potential are originated from the ones in (7); D_{inv} is a descendant of G_{inv} from (7): to go over from G_{inv} to D_{inv} one should make all pairing of non-soft photons in G_{inv} and all the reductions of space-time coordinats which was involved in going over from the G_β of (1) to the D_β of (17); at last, all gauge potentials in (35) are taken at zero of space coordinate in accord with the $\delta(\mathbf{x})$ of eq.(17). The difference between the lhs and the rhs of eq.(35) does not conribute to the term under the calculation.

The leading in the nonrelativistic approximation contribution to D_{inv} is the same as for D_β — it is just the propagator of the nonrelativistic Coulomb problem. We explicitly calculate the leading contribution to the dependence of $D_{inv}(t, A)$ on the gauge potential in its expansion over soft momenta of the external photons. Not surprisingly, the dipole interaction of the pair with the external photon field arise in this approximation:

$$D_{inv}(t, A) \approx \left(i\frac{\partial}{\partial t} - H_c + e\mathbf{r}\mathcal{E}(t) \right)^{-1}, \tag{36}$$

where H_c is the hamiltonian of the nonrelativistic Coulomb problem and $\mathcal{E}$ is the strength of the electric field:

$$\mathcal{E}(t) \equiv -\dot{\mathbf{A}}(t) + \nabla A_0(t). \tag{37}$$

Substituting (36) in (35) and keeping terms with only one soft photon propagator we get expressions which sum contains the term under calculation:

$$e^2 \left(L_s \mathbf{r}\mathbf{A}(t) D_{nr}(t) \mathbf{r}\mathbf{A}(0) \right)_{A=0}, \tag{38}$$

$$e^2 \left(L_s \int d\tau_1 d\tau_2 \, D_{nr}(t-\tau_1)\mathbf{r}\mathcal{E}(\tau_1) D_{nr}(\tau_1-\tau_2)\mathbf{r}\mathcal{E}(\tau_2) D_{nr}(\tau_2) \right)_{A=0}, \tag{39}$$

$$ie^2 \Big(L_s \int d\tau \, \big(D_{nr}(t-\tau)\mathbf{r}\mathcal{E}(\tau) D_{nr}(\tau)\mathbf{r}\mathbf{A}(0) \; - \tag{40}$$

$$\mathbf{r}\mathbf{A}(t) D_{nr}(t-\tau)\mathbf{r}\mathcal{E}(\tau) D_{nr}(\tau) \big) \Big)_{A=0},$$

where $D_{nr}(t)$ is the propagator of the nonrelativistic Coulomb problem from the rhs of eq.(18).

The next step is to pick out a contribution of a level E_0 in (38),(39),(40). That is achievable by the replacement

$$D_{nr}(t) \to e^{-iE_0 t}\theta(t) P(E_0). \tag{41}$$

The last ingredient that one needs to calculate (38),(39),(40) is the time dependence of the soft photon propagators. It can be deduced from (2) as

$$\begin{aligned} (L_s A_i(t_1) A_j(t_2)) &= \theta\left((t_1-t_2)^2 > t_c^2 \right) \frac{\delta_{ij}\left(-1+\frac{\beta}{2}\right)}{4\pi^2 (t_1-t_2)^2}, \\ (L_s A_i(t_1) \mathcal{E}_j(t_2)) &= \theta\left((t_1-t_2)^2 > t_c^2 \right) \frac{\delta_{ij}}{2\pi^2 (t_1-t_2)^3}, \\ (L_s \mathcal{E}_i(t_1) \mathcal{E}_j(t_2)) &= \theta\left((t_1-t_2)^2 > t_c^2 \right) \frac{\delta_{ij}}{\pi^2 (t_1-t_2)^4}. \end{aligned} \tag{42}$$

Here the θ-functions are to account for the softness of the participating photons ($t_c \sim 1/E_0$).

Taking (42) into account we get the following contributions from (38),(39),(40):

$$
\begin{aligned}
(38) \quad &\rightarrow \quad \frac{1}{t^2}\theta(t)e^{-iE_0t}\frac{\alpha}{\pi}\left(-1+\frac{\beta}{2}\right)\mathbf{r}P(E_0)\mathbf{r}, \\
(39) \quad &\rightarrow \quad \frac{1}{t^2}\theta(t)e^{-iE_0t}\frac{\alpha}{\pi}\frac{2}{3}P(E_0)\mathbf{r}P(E_0)\mathbf{r}P(E_0), \\
(40) \quad &\rightarrow \quad \frac{1}{t^2}\theta(t)e^{-iE_0t}\frac{\alpha}{\pi}i\left(P(E_0)\mathbf{r}P(E_0)\mathbf{r} - \mathbf{r}P(E_0)\mathbf{r}P(E_0)\right).
\end{aligned} \tag{43}
$$

The sum of the above terms yields the result of our calculation:

$$
\begin{aligned}
\Sigma_\beta(E_0) \quad = \quad & \frac{\alpha}{\pi}\Big(\frac{2}{3}P(E_0)\mathbf{r}P(E_0)\mathbf{r}P(E_0) + (-1+\frac{\beta}{2})\mathbf{r}P(E_0)\mathbf{r} + \\
& i(P(E_0)\mathbf{r}P(E_0)\mathbf{r} - \mathbf{r}P(E_0)\mathbf{r}P(E_0))\Big).
\end{aligned} \tag{44}
$$

One can explicitly check that β-dependence of $\Sigma_\beta(E_0)$ is the right one — i.e. if one subtracts the Σ-term from the propagator before the definition of the energy shift operator, the latter becomes gauge independent. Another observation is that the Σ-term cannot be killed by any choice of the gauge (in contrast to the case of charged fermion propagator where an analogous term is equal to zero in the Yennie gauge).

Summing up, in this paper we derived a relation between QED Green functions of different gauges. We used it to check the gauge invariance of the energy shift operator. It turns out to be gauge dependent. This fact forced us to recognize that energy shifts are not one, and the only one, source for the positive powers of time near the oscillating exponentials in the propagator of the pair. We found a particular additional source of the positive powers of time which is responsible for the gauge dependence of the naive energy shift operator. We conclude by an observation that at the moment we have not a clear definition of the energy shift operator — to get it one needs a criterion for picking out contributions to the positive powers of time originating from the energy shifts.

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References

[1] C. I. Westbrook *et al.*, *Phys.Rev.Lett.* **58**(1987)1328

[2] P. Labelle, G. P. Lepage, U. Magnea, *Order $m\alpha^8$ contribution to the decay rate of Orthopositronium*, preprint CLNS/93/1199, 1993, hep-ph 9310208

[3] I. B. Khriplovich, A. I. Milstein, *JETP* **79**(1994)379

[4] W. E. Caswell, G. P. Lepage, *Phys. Rev.* **A18**(1978)810

[5] R. Barbieri, E. Remiddi, *Nucl. Phys.* **B141**(1978)417

[6] O. Steinman, *Nucl. Phys.* **B119**(1982)394

[7] G. S. Adkins, *Ann. Phys. (N.Y.)* **146**(1983)78

[8] A. N. Vassiliev, *Functional Methods in Quantum Field Theory and Statistical Mechanics* (L.S.U., Leningrad, 1976);in Russian

[9] G. B. Pivovarov,*Improved Nonrelativistic QED and other Effective Field Theories*, in "Quarks-94", eds. V. A. Matveev *et al.* (World Scientific), in press

[10] V. B. Beresteskii, E. M. Lifshits, L. P. Pitaevskii, *Quantum Electrodinamics* (Nauka, Moscow, 1980); in Russian